



THE UNIVERSITY *of* EDINBURGH

Edinburgh Research Explorer

Diffraction induced instability of coupled dark solitary waves

Citation for published version:

Assanto, G, MacNeil, JML & Smyth, N 2015, 'Diffraction induced instability of coupled dark solitary waves', *Optics Letters*, vol. 40, no. 8, pp. 1771-1774. <https://doi.org/10.1364/OL.40.001771>

Digital Object Identifier (DOI):

[10.1364/OL.40.001771](https://doi.org/10.1364/OL.40.001771)

Link:

[Link to publication record in Edinburgh Research Explorer](#)

Document Version:

Peer reviewed version

Published In:

Optics Letters

General rights

Copyright for the publications made accessible via the Edinburgh Research Explorer is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

Take down policy

The University of Edinburgh has made every reasonable effort to ensure that Edinburgh Research Explorer content complies with UK legislation. If you believe that the public display of this file breaches copyright please contact openaccess@ed.ac.uk providing details, and we will remove access to the work immediately and investigate your claim.



Diffraction induced instability of coupled dark solitary waves

Gaetano Assanto,¹ J. Michael L. MacNeil,² and Noel F. Smyth²

¹*NooEL– Nonlinear Optics and OptoElectronics Lab, University “Roma Tre”, 00146 Rome, Italy
and*

The University of Wollongong, Wollongong, 2522 NSW, Australia

²*School of Mathematics, University of Edinburgh,
Edinburgh EH9 3JZ, Scotland, U.K.*

We report on a novel instability arising from the propagation of coupled dark solitary beams governed by coupled defocusing nonlinear Schrödinger equations. Considering dark notches on backgrounds with different wavelengths, hence different diffraction coefficients, we find that the vector dark soliton solution is unstable to radiation modes. Using perturbation theory and numerical integration, we demonstrate the component undergoing stronger diffraction radiates away, leaving a single dark soliton in the other mode/wavelength.

PACS numbers: 42.65.Tg, 05.45.Yv

Wave instabilities arising from nonlinear wave models are among the most intriguing of physical phenomena. The discovery of modulational instability (MI) by Benjamin and Fier [1–3] governed by the nonlinear Schrödinger (NLS) equation demonstrated a remarkable mechanism that has no analog in linear systems. The NLS equation is a standard, ubiquitous equation in nonlinear wave theory as it describes weakly nonlinear wave packets with a narrow spectrum and related effects [2, 4], including instabilities and solitary waves in nonlinear optics [2, 5], as well as fluid mechanics [2, 3] and biology [6]. In addition to systems governed by the NLS equation, MI and other instabilities have been discovered for a wide array of generally applicable integrable and non-integrable equations, including systems applicable to nonlocal, saturable, dissipative and higher order nonlinear optics. Collectively, the broad application of these integrable and non-integrable wave systems makes the study of their associated instabilities one of fundamental importance in optics, and in physics in general.

The focusing NLS equation models beam self-focusing and has bright soliton solutions, i.e. humps on a zero background. The defocusing NLS equation models beam self-defocusing and supports dark soliton solutions, i.e. dips (notches) in a non-zero background of constant amplitude, with a π phase change of the electric field on axis [5, 7]. Several demonstrations of individual and coupled dark solitary waves have been reported in various media [8], including semiconductors [9], soft matter [10–13], photovoltaic photorefractive crystals [14, 15] and metamaterials [16]. Here, we deal with vector solitary wave solutions of two coupled defocusing NLS equations, as, for example, governing the incoherent interaction of two dark optical beams in a generic nonlinear dielectric [5]. Previous work has shown that stable vector dark solitary wave solutions of such equations exist if the diffraction coefficients are equal [5, 17, 18].

In the present work, we find a novel instability, driven by a difference in diffraction coefficients, of dark vector solitary waves. If the diffraction of the two components (dark beams) is different, for example beams of different

wavelengths [19] or polarizations, the coupled vector solitary wave is unstable to radiation. One of the wavepackets collapses into radiation, leaving a single dark solitary wave in the other mode. We investigate this novel radiation induced instability using a perturbation theory analysis and numerical solutions.

Let us consider a system of two coupled, defocusing NLS equations governing the collinear co-propagation of two incoherent dark beams in a Kerr medium with a self-defocusing optical response. Each of the two dark modes consists of a notch on a constant background, with a zero electric field on axis and distinct wavelengths, so they experience different phase velocities and diffraction. The coupled, defocusing NLS equations governing these two dark beams of amplitudes u and v are then

$$\begin{aligned} i \frac{\partial u}{\partial z} + \frac{D_u}{2} \frac{\partial^2 u}{\partial x^2} - (|u|^2 + |v|^2) u &= 0, \\ i \frac{\partial v}{\partial z} + \frac{D_v}{2} \frac{\partial^2 v}{\partial x^2} - (|u|^2 + |v|^2) v &= 0, \end{aligned} \quad (1)$$

with D_u and D_v the diffraction coefficients for modes u and v , respectively. Such equations arise in various areas of nonlinear optics in general [5] and nonlinear two-colour beam propagation in, e.g., reorientational soft-matter [13, 18, 19] and photorefractive crystals [14, 15]. Such equations arise in various areas of nonlinear optics in general [5] and nonlinear two colour beam propagation in, for example, reorientational soft matter [19] and photorefractive crystals [14, 15]. In the limit of equal diffraction coefficients $D_u = D_v$, the focusing equivalent of system (1) is the integrable Manakov system [20].

To understand the evolution of the coupled dark solitary waves governed by the coupled NLS equations (1), we first consider the limit of the diffraction coefficients differing by a small amount, so that $D_v = D_u + \epsilon$, where $|\epsilon| \ll D_u$. This case usefully demonstrates that the instability of the two coupled dark solitary waves is due to the rise of secular terms in a regular perturbation expansion. We further assume that the dark solitons in the two modes have the same constant background level

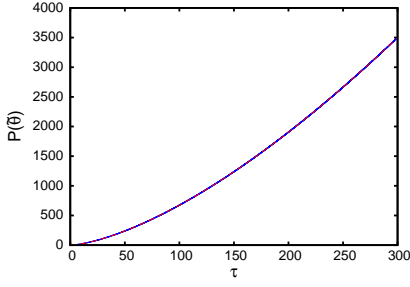


FIG. 1: (Color online) Power P of the difference $\tilde{\theta} = u(\tilde{x}, \tau) - v(\tilde{x}, \tau)$ between the two dark beams. $\tilde{\theta} = 0$ at $z = 0$. Blue (solid) line: numerically calculated power; red (dashed) line: fit.

$U_0 = V_0$, which is of more physical interest. The steady dark soliton solutions in the modes u and v can then be expanded as

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \dots, \quad v = v_0 + \epsilon v_1 + \epsilon^2 v_2 + \dots \quad (2)$$

At first order the dark soliton solution is [5]

$$u_0 = v_0 = U_0 \tanh(\gamma x) e^{-2iU_0^2 z} \quad (3)$$

with $\gamma = \sqrt{2}U_0/\sqrt{D_u}$. At second order, $O(\epsilon)$, the corrections to this steady dark soliton are determined by

$$i \frac{\partial u_1}{\partial z} + \frac{D_u}{2} \frac{\partial^2 u_1}{\partial x^2} - |u_0|^2 (3u_1 + v_1) - u_0^2 (u_1^* + v_1^*) = 0, \quad (4)$$

$$i \frac{\partial v_1}{\partial z} + \frac{D_u}{2} \frac{\partial^2 v_1}{\partial x^2} - |u_0|^2 (3v_1 + u_1) - u_0^2 (u_1^* + v_1^*) = -\frac{1}{2} \frac{\partial^2 u_0}{\partial x^2}, \quad (5)$$

where the superscript $*$ denotes the complex conjugate. Subtracting these equations gives that the difference $\theta = u_1 - v_1$ is governed by the forced Schrödinger equation

$$i \frac{\partial \theta}{\partial z} + \frac{D_u}{2} \frac{\partial^2 \theta}{\partial x^2} - 2|u_0|^2 \theta = \frac{1}{2} \frac{\partial^2 u_0}{\partial x^2}. \quad (6)$$

We first remark that the $\partial^2 u_0 / \partial x^2$ forcing term of the Schrödinger equation (6) is secular in x as it is proportional to $\text{sech}^2 \gamma x \tanh \gamma x = \tanh \gamma x - \tanh^3 \gamma x$, with $\tanh \gamma x$ being a solution of the homogeneous equation. This can be seen in further detail by seeking solitary wave solutions of the form

$$\theta = \Theta(x) e^{-2iU_0^2 z} \quad (7)$$

with Θ real, so that the forced Schrödinger equation (6) becomes

$$\frac{D_u}{2} \Theta'' + 2U_0^2 \text{sech}^2(\gamma x) \Theta = -\gamma^2 U_0 \text{sech}^2(\gamma x) \tanh(\gamma x). \quad (8)$$

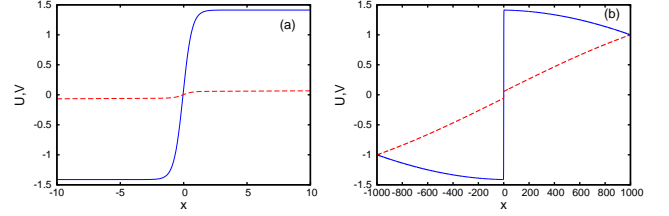


FIG. 2: (Color online) Numerically computed coupled dark solitary waves with artificial Dirichlet conditions. u mode: blue (solid) line; v mode: red (dotted) line. The diffraction parameters are $D_u = 1.0$ and $D_v = 1.02$. (a) Local view, (b) expanded view.

As $x \rightarrow \infty$, the particular solution of this equation tends to

$$\Theta \sim \frac{\gamma}{D_u} U_0 x (1 - 2e^{-2\gamma x}) + \dots, \quad (9)$$

with a symmetric asymptotic trend as $x \rightarrow -\infty$. The perturbation expansion then breaks down at $O(\epsilon)$ and there are no localized vector dark solitary waves of the coupled defocusing NLS equations (1).

This asymptotic result in the limit of a small difference in diffraction coefficients was verified using full z dependent numerical solutions. These numerical solutions also show that the asymptotic result for the non-existence of coupled dark solitary waves carry over to a finite difference in diffraction coefficients. In addition, they show in detail the dynamic collapse of the coupled state. The forced Schrödinger equation (6) was solved numerically using a 4th-order Runge-Kutta scheme (RK4) in the normalized unit $\tau = U_\infty^2 z$ and centered finite differences in $\tilde{x} = \gamma x$ with $\theta(x, z) = \tilde{\theta}(\tilde{x}, \tau)$, so that the simulations are independent of the background U_0 and the diffraction coefficient D_u . Fig. 1 illustrates clear, substantial growth in the power P of θ , defined as the integral of $|\tilde{\theta}|^2$ (obtained numerically), demonstrating secular growth is not solely confined to the spatial dimension, but the evolution variable as well. These numerical results were fitted with the curve $f(\tau) = 0.6727\tau^{\frac{3}{2}}$. This power growth can be obtained from an asymptotic analysis of (6) for large z . We set $\theta = i \int_0^z \Phi(\tilde{x}, \tilde{s}) d\tilde{s} e^{-2iU_0^2 z}$, where Φ is the solution of

$$i \frac{\partial \Phi}{\partial \tilde{z}} + \frac{\partial^2 \Phi}{\partial \tilde{x}^2} + 2 \text{sech}^2 \tilde{x} \Phi = 0, \quad \Phi(x, 0) = \gamma^2 \text{sech}^2 \tilde{x} \tanh \tilde{x} \quad (10)$$

Now $\max_x |\Phi| \sim Cz^{-1/2}$ [22]. As for large z it can be shown that (6) has a similarity solution of the form $f(x/\sqrt{z})e^{-2iU_0^2 z}$, we then have that $\int_{-\infty}^{\infty} |\theta|^2 dx \sim \tilde{C}z^{\frac{3}{2}}$.

This resonant instability due to the slightly different diffraction coefficients of the two modes does not occur for the corresponding coupled focusing NLS equations, for which the negative signs in Eqns. (1) are replaced by positive signs. In this case, the corresponding equation for Θ_b (subscript for “bright”) for coupled solitary waves

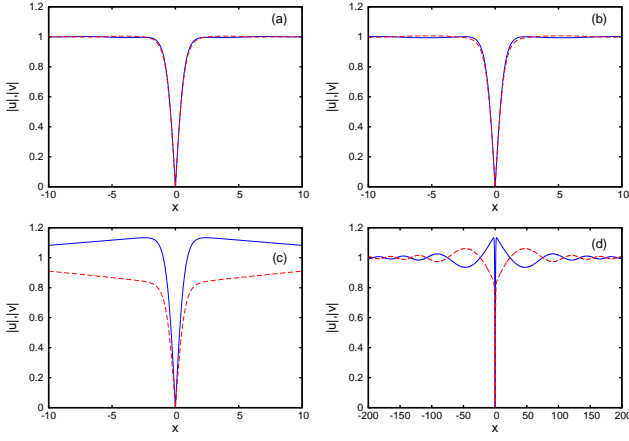


FIG. 3: (Color online) Evolution of the instability for a small difference in the diffraction coefficients. u beam: blue (solid) line; v beam: red (dotted) line. (a) Profiles at $z = 3$, (b) profiles at $z = 10$, (c) solution around $x = 0$ at $z = 700$, (d) expanded solution at $z = 700$. $U_0 = V_0 = 1$ and the diffraction parameters are $D_u = 1.0$ and $D_v = 1.02$.

$u_0 = v_0 = a \operatorname{sech} \tilde{\gamma} x e^{ia^2 z}$ is

$$\begin{aligned} \frac{D_u}{2} \Theta_b'' + a^2 (2 \operatorname{sech}^2 \tilde{\gamma} x - 1) \Theta_b \\ = \frac{a \tilde{\gamma}^2}{2} (\operatorname{sech} \tilde{\gamma} x - 2 \operatorname{sech}^3 \tilde{\gamma} x), \end{aligned} \quad (11)$$

with $\tilde{\gamma} = \sqrt{2}a/\sqrt{D_u}$. The component $\operatorname{sech} \tilde{\gamma} x$ of the forcing is still resonant, but the $-a^2 \Theta_b$ term in the homogeneous equation implies that the particular solution is of the form $x e^{-\tilde{\gamma} x}$ as $x \rightarrow \infty$, and so does not grow. Therefore, stable vector bright solitary wave solutions of two coupled focusing NLS equations exist even when the diffraction coefficients are different.

The lack of a steady, coupled, localized state was confirmed by full numerical solutions. We first integrated the governing equations (1) using the same RK4 z -stepping and centered differences for the x derivatives as used in numerically integrating (6). The domain was taken very large ($L \sim 2000$) to avoid boundary effects such as radiation reflection. In each simulation, the quantities $u \exp(-i(U_0^2 + V_0^2)z)$ and $v \exp(-i(U_0^2 + V_0^2)z)$ were found to be purely real for arbitrary stopping “times” z . We therefore sought numerical solutions for steady coupled solitary waves $u = U(x)e^{-i\sigma z}$ and $v = V(x)e^{-i\sigma z}$, so that

$$\begin{aligned} \frac{D_u}{2} \frac{\partial^2 U}{\partial x^2} + \sigma U - (U^2 + V^2) U &= 0, \\ \frac{D_v}{2} \frac{\partial^2 V}{\partial x^2} + \sigma V - (U^2 + V^2) V &= 0 \end{aligned} \quad (12)$$

for real U and V . The system (12) was solved using a Newton iteration scheme (see [21] for details) with the exact solution for $D_u = D_v$ as an initial guess and continuing, without loss of generality, the D_v parameter so

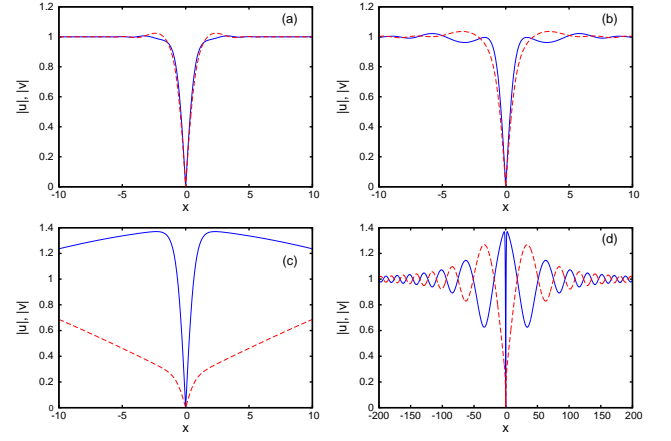


FIG. 4: (Color online) Evolution of the instability for a larger difference in the diffraction coefficients. u beam: blue (solid) line; v beam: red (dotted) line. (a) Profiles at $z = 1$, (b) profiles at $z = 3$, (c) solution around $x = 0$ at $z = 500$, (d) expanded solution at $z = 500$. $U_0 = V_0 = 1$ and the diffraction constants are $D_u = 1.0$ and $D_v = 1.25$.

that it became larger than D_u . Both fixed boundary conditions, so that $U(L) = U_0$ and $U(-L) = -U_0$ and similarly for V with $\sigma = U_0^2 + V_0^2$, and zero flux boundary conditions, so that $\partial_x U(\pm L) = \partial_x V(\pm L) = 0$, were applied to determine whether the choice of boundary condition made any difference to the stability. A representative example for solutions found with the fixed boundary conditions is shown in Fig. 2. Fig. 2(a) shows that locally, around $x = 0$, the v dark beam is the trivial solution, while the u mode corresponds to the exact solitary wave with $\sigma = \sqrt{2}U_0$. Fig. 2(b) shows that, away from $x = 0$, the dark modes approach the original background levels. This is due to the fixed boundary conditions at $x = \pm L$. This figure clearly shows qualitatively that the difference $U - V$ matches the linear growth behaviour predicted by the asymptotic result (9) and is a behaviour found in full numerical simulations, discussed later. In the case of zero flux boundary conditions, the scheme converged to the steady state corresponding to the exact dark NLS soliton solution with $v = 0$, in agreement with the fixed boundary condition result. These numerical results indicate that a localized steady state consists solely of a single mode, as opposed to coupled modes found in the focussing NLS case.

Full numerical solutions of equations (1) confirm those found from the study of the steady states. Figure 3 illustrates full numerical solutions of the coupled defocusing NLS equations (1) for a small difference in diffraction coefficients, $D_u = 1.0$ and $D_v = 1.02$, as for the perturbation solution discussed previously. The initial conditions used were

$$u = v = U_0 \tanh(\gamma x), \quad (13)$$

with $\gamma = \sqrt{2}U_0/\sqrt{D_u}$. Clearly the u beam is settling down to a local dark solitary wave, while the v beam is

spreading out and decaying in a way that approaches, locally, the steady solutions found with the fixed boundary conditions. This instability is driven by the difference in the diffraction coefficients and occurs via a non-standard process. The larger diffraction D_v of the v mode makes it initially diffract more than the u dark beam. Such widening of the v notch relative to the u mode causes the u dark beam to deform, which reinforces the widening of v . The latter is accompanied by the shedding of diffractive radiation of growing amplitude, with the v mode progressively decaying to zero, as shown in Fig. 3(c) in the vicinity of $x = 0$ and in Fig. 3(d) over a larger region around $x = 0$ in order to emphasise the shed diffractive radiation. The decay of v to 0 is accompanied by the growth of shed radiation as both the u and v modes individually conserve power, i.e. “mass” in the sense of invariances of the Lagrangian of the coupled system (1); the power $P_u = \int_{-\infty}^{\infty} (|U_0|^2 - |u|^2) dx$ of the u mode and the power $P_v = \int_{-\infty}^{\infty} (|V_0|^2 - |v|^2) dx$ of the v mode are individually conserved. To conserve P_v , however, the v mode can only decay by shedding radiation rising above $v = V_0$ and so balance the decay to 0 in a region expanding from $x = 0$. Therefore, the released diffractive radiation increases in amplitude. Fig. 3(c) shows that, locally, the v -component decay is accompanied by the u mode evolving to the new background level $\sqrt{2}U_0$ pre-

dicted by the numerical steady state results.

This instability mechanism of two coupled dark solitary waves is considerably more pronounced with an increased difference in the diffraction coefficients of the two modes, as illustrated in Fig. 4, for which $D_u = 1.0$ and $D_v = 1.25$. The instability now evolves on a shorter z scale, as expected. Again, Fig. 4(c) shows that the v dark beam decays and that the u mode moves to the new background $\sqrt{2}U_0$. This further confirms the lack of a localized coupled steady state, the main conclusion of the present work.

We studied two coupled defocusing NLS equations describing two incoherent dark beams propagating collinearly in a Kerr medium. We found that this system does not possess a stable coupled (vector) solitary wave solution if the two dark components undergo different diffraction, as one mode sheds radiation and progressively decays, while the other settles to a dark soliton. This behavior is due to a resonant instability induced by the difference in diffraction coefficients. Numerical solutions fully confirm this unstable evolution for arbitrary differences in diffraction constants.

This work was supported by the Royal Society of London under grant IE111560. GA thanks Prof. T. Marchant for his fruitful visit at the University of Wollongong.

-
- [1] T.B. Benjamin and J.E. Feir, *J. Fluid Mech.*, **27**, 417–430 (1967).
 - [2] G.B. Whitham, *Linear and Nonlinear Waves*, J. Wiley and Sons, New York (1974).
 - [3] G.L. Lamb, *Elements of Soliton Theory*, J. Wiley and Sons, New York (1980).
 - [4] A. C. Newell, *Solitons in Mathematics and Physics*, SIAM, Philadelphia (1985).
 - [5] Y.S. Kivshar and G.P. Agrawal, *Optical Solitons. From Fibers to Photonic Crystals*, Academic Press, San Diego (2003).
 - [6] A. S. Davydov, *Solitons in Molecular Systems*, second edition, Kluwer Acad. Publ., Dordrecht, Boston (1991).
 - [7] Y.S. Kivshar, *IEEE J. Quantum Electron.* **29**, 250–264 (1993).
 - [8] G. A. Swartzlander, Jr., D. R. Andersen, J. J. Regan, H. Yin, and A. E. Kaplan, *Phys. Rev. Lett.* **66**, 1583 (1991).
 - [9] G. R. Allen, S. R. Skinner, D. R. Andersen, and A. L. Smirl, *Opt. Lett.* **16**, 156–158 (1991).
 - [10] B. Luther-Davies and Y. Xiaoping, *Opt. Lett.* **17**, 496–498 (1992).
 - [11] M. Shalaby and A.C. Barthelemy, *IEEE J. Quantum Electron.* **28**, 2736–2741 (1992).
 - [12] A. Dreischuh, D. N. Neshev, D. E. Petersen, O. Bang and W. Krolikowski, *Phys. Rev. Lett.* **96**, 043901 (2006).
 - [13] A. Piccardi, A. Alberucci, N. Tabiryan and G. Assanto, *Opt. Lett.* **36**, 1356–1358 (2011).
 - [14] M. Taya, M. C. Bashaw, M. M. Fejer, M. Segev and G. C. Valley, *Phys. Rev. A* **52**, 3095–3100 (1995).
 - [15] Z. Chen, M. Mitchell, M.-F. Shih, M. Segev, M. H. Garrett, and G. C. Valley, *Opt. Lett.* **21**, 629–631 (1996).
 - [16] M. Tlidi, P. Kockaert and L. Gelens, *Phys. Rev. A*, **84**, 013807 (2011).
 - [17] Y. S. Kivshar and S.K. Turitsyn, *Opt. Lett.* **18**, 337–339 (1993).
 - [18] Y. S. Kivshar and B. Luther-Davies, *Phys. Rep.* **298**, 81–197 (1998).
 - [19] A. Alberucci, M. Peccianti, G. Assanto, A. Dyadyusha and M. Kaczmarek, *Phys. Rev. Lett.* **97**, 153903 (2006).
 - [20] S. V. Manakov, *Zh. Eksp. Teor. Fiz.* **65**, 505 (1973) [*Sov. Phys. JETP* **38**, 248–253 (1974)].
 - [21] J. Yang, *J. Comp. Phys.* **228**, 7007–7024 (2009).
 - [22] W. Schlag, *Mathematical Aspects of Nonlinear Dispersive Equations*, 255–286 (2007).

- [1]. T.B. Benjamin and J.E. Feir, “The disintegration of wave trains on deep water. Part 1. Theory,” *J. Fluid Mech.*, **27**, 417–430 (1967).
- [2]. G.B. Whitham, *Linear and Nonlinear Waves*, J. Wiley and Sons, New York (1974).
- [3]. G.L. Lamb, *Elements of Soliton Theory*, J. Wiley and Sons, New York (1980).
- [4]. A. C. Newell, *Solitons in Mathematics and Physics*, SIAM, Philadelphia (1985).
- [5]. Y.S. Kivshar and G.P. Agrawal, *Optical Solitons. From Fibers to Photonic Crystals*, Academic Press, San Diego (2003).
- [6]. A. S. Davydov, *Solitons in Molecular Systems*, second edition, Kluwer Acad. Publ., Dordrecht, Boston (1991).
- [7]. Y.S. Kivshar, “Dark solitons in nonlinear optics,” *IEEE J. Quantum Electron.* **29**, 250–264 (1993)
- [8]. G. A. Swartzlander, Jr., D. R. Andersen, J. J. Regan, H. Yin, and A. E. Kaplan, “Spatial dark-soliton stripes and grids in self-defocusing materials,” *Phys. Rev. Lett.* **66**, 1583 (1991).
- [9]. G. R. Allen, S. R. Skinner, D. R. Andersen, and A. L. Smirl, “Observation of fundamental dark spatial solitons in semiconductors using picosecond pulses,” *Opt. Lett.* **16**, 156–158 (1991).
- [10]. B. Luther-Davies and Y. Xiaoping, “Waveguides and Y junctions formed in bulk media by using dark spatial solitons,” *Opt. Lett.* **17**, 496–498 (1992).
- [11]. M. Shalaby and A.C. Barthelemy, “Observation of the self-guided propagation of a dark and bright spatial soliton pair in a focusing nonlinear medium,” *IEEE J. Quantum Electron.* **28**, 2736–2741 (1992).
- [12]. A. Dreischuh, D. N. Neshev, D. E. Petersen, O. Bang and W. Krolikowski, “Observation of attraction between dark solitons,” *Phys. Rev. Lett.* **96**, 043901 (2006)
- [13]. A. Piccardi, A. Alberucci, N. Tabiryan and G. Assanto, “Dark nematons,” *Opt. Lett.* **36**, 1356–1358 (2011).
- [14]. M. Taya, M. C. Bashaw, M. M. Fejer, M. Segev and G. C. Valley, “Observation of dark photovoltaic spatial solitons,” *Phys. Rev. A* **52**, 3095–3100 (1995).
- [15]. Z. Chen, M. Mitchell, M.-F. Shih, M. Segev, M. H. Garrett, and G. C. Valley, “Steady-state dark photorefractive screening solitons,” *Opt. Lett.* **21**, 629–631 (1996).
- [16]. M. Tlidi, P. Kockaert and L. Gelens, “Dark localised structures in a cavity filled with a left-handed material,” *Phys. Rev. A*, **84**, 013807 (2011).
- [17]. Y. S. Kivshar and S.K. Turitsyn, “Vector dark solitons,” *Opt. Lett.* **18**, 337–339 (1993).
- [18]. Y. S. Kivshar and B. Luther-Davies, “Dark optical solitons: physics and applications,” *Phys. Rep.* **298**, 81–197 (1998).
- [19]. A. Alberucci, M. Peccianti, G. Assanto, A. Dyadyusha and M. Kaczmarek, “Two-color vector solitons in nonlocal media,” *Phys. Rev. Lett.* **97**, 153903 (2006).
- [20]. S. V. Manakov, “On the theory of two-dimensional stationary self-focusing of electromagnetic waves,” *Zh. Eksp. Teor. Fiz.* **65**, 505 (1973) [*Sov. Phys. JETP* **38**, 248–253 (1974)].
- [21]. J. Yang, “Newton-conjugate-gradient methods for solitary wave computations,” *J. Comp. Phys.* **228**, 7007–7024 (2009).
- [22]. W. Schlag, “Dispersive Decay Estimates for Schrödinger operators: a survey,” in *Mathematical Aspects of Nonlinear Dispersive Equations*, 255–286 (2007).